

Nonlinear pseudo-differential equations defined by elliptic symbols on $L^p(\mathbb{R}^n)$ and the fractional Laplacian

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Abstract

We develop an $L^p(\mathbb{R}^n)$ -functional calculus appropriated for interpreting “non-classical symbols” of the form $a(-\Delta)$, and for proving existence in $L^q(\mathbb{R}^n)$, some $q > p$, of solutions to nonlinear pseudo-differential equations of the form $[1 + a(-\Delta)]^{s/2}(u) = V(\cdot, u)$. More precisely, we use the theory of Fourier multipliers for constructing suitable domains on which the formal operator appearing in the above equation can be rigorously defined, and we prove existence of solutions belonging to these domains. We also include applications of the theory to equations of physical interest involving the fractional Laplace operator such as the Allen-Cahn equation.

1 Introduction

The aim of this work is to study nonlinear equations of the form

$$[1 + a(-\Delta)]^{s/2}(u) = V(\cdot, u) , \tag{1}$$

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in which Δ stands for the Laplace operator on \mathbb{R}^n and the function $a(t)$, $t \geq 0$, is a “non-classical symbol” of elliptic type, see Definition 2.1 below. Two of the present authors, in collaboration with P. Górká, have developed a mathematical framework which allows us to understand equations of the form (1) in the $L^2(\mathbb{R}^n)$ context, see [17, 18, 19]. In this article we consider these equations in the completely general context of Lebesgue spaces. We show that, as is the case for classical elliptic equations, we can prove existence of solutions to (1) with integrability and differentiability properties that go beyond the ones we would expect for $L^p(\mathbb{R}^n)$ functions. We recall that equations such as (1) belong to a class of pseudo-differential equations which have been treated in a formal manner as “equations in infinitely many derivatives” in the Physics literature, see for instance [3, 4, 5, 10, 11, 33]. These also include non-local equations, as observed recently in [12], and equations of physical interest depending on fractional Laplace operators, see [8, 9, 13, 15, 16, 27].

It is very important to stress the fact that the approach we followed in [17, 18, 19] does not apply in an obvious way to the $L^p(\mathbb{R}^n)$ framework. To give a basic example, we note that some of the arguments used in [17, 18, 19] break down because of the non-existence of a Plancherel theorem in $L^p(\mathbb{R}^n)$ (or, in other words because the Fourier transform is not a unitary operator on $L^p(\mathbb{R}^n)$). Nevertheless, in this paper we show that it is possible to develop alternative techniques to deal with equations such as (1) in general Lebesgue spaces.

Our main tool is the use of Fourier multipliers after [2, 28]. These multipliers allow us to introduce a scale of spaces $\mathcal{H}^{s,p}(a)$ sitting inside $L^p(\mathbb{R}^n)$, which are determined by the symbol a . Partially motivated by Lions’ classical paper [23], we look at some embedding properties of the spaces $\mathcal{H}^{s,p}(a)$, and we also consider their relation with fractional Sobolev spaces. In particular, in the case of $L^p(\mathbb{R}^n)$ spaces of radial functions, we obtain a “radial scale” of spaces $\mathcal{H}_r^{s,p}(a)$, see Section 4 below. The results of [23] imply that $\mathcal{H}_r^{s,p}(a)$ is compactly embedded into an appropriate $L^p(\mathbb{R}^n)$.

We organize our work as follows. In Section 2 we introduce the class \mathcal{G}_s^β of symbols we consider thereafter, see Definition 2.1. An important example of allowable symbol is the fractional Laplacian, as we show in Lemma 2.2. In Section 3 we introduce Fourier multipliers and we develop a functional calculus in $L^p(\mathbb{R}^n)$ for \mathcal{G}_s^β symbols a . This functional calculus takes into account the identification of the operator $a(-\Delta)$ with the symbol $a(|\xi|^2)$, motivated by the well-known fact that in the $L^2(\mathbb{R}^n)$ setting this correspondence is rigorously achieved with the aid of the Fourier transform. We then define a bona fide operator with domain $\mathcal{H}^{s,p}(a) \subseteq L^p(\mathbb{R}^n)$ which corresponds to the formal operator appearing on the left hand side of Equation (1).

The embedding properties mentioned in the previous paragraph are also considered in this section. In Section 4 we consider Equation (1) and we prove two existence theorems: the first in a “localized” setting, and the second in the setting of functions with radial symmetry, see Theorems 4.3 and 4.5 respectively. The aforementioned compact embedding theorems appearing in [23] are crucially used in the proof of Theorem 4.5. Finally, in Section 5 we apply our radial existence result to prove $L^p(\mathbb{R})$ -existence theorems for equations of physical interest involving the fractional Laplace operator. Interestingly, the solutions we find belong to one of the spaces $\mathcal{H}^{s,p}(a) \subseteq L^p(\mathbb{R}^n)$ defined in Section 2; since we have a “radial scale” of spaces, we can actually prove that our solutions belong to $L^{\delta p}(\mathbb{R})$ with $\delta > 1$. We have chosen as examples the (perturbed) fractional Allen-Cahn equation, the Benjamin-Ono equation, and a fractional non-linear Schrödinger equation.

2 Preliminaries

In this section we establish some basic notations, and we collect some technical results which will be used thereafter.

Let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and let α be a multi-index with n slots, i.e., $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, where $\alpha_i \in \mathbb{Z}$, $\alpha_i \geq 0$. We write x^α for $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$. The length of the multi-index α is the positive integer $|\alpha| = \sum_{i=1}^n \alpha_i$. If α and β are multi-indexes, we say that $\alpha \leq \beta$ if $\alpha_i \leq \beta_i$ for all $i = 1, \dots, n$.

For an arbitrary multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$, the differential operator D^α is defined as

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

Let f be a real-valued function $f \in C^k(\mathbb{R})$, and let $\gamma = (\gamma_1, \dots, \gamma_n)$ be a multi-index with k non-zero components. We denote by f^γ the real-valued function defined by

$$f^\gamma(t) = (f'(t))^{\gamma_{i_1}} (f''(t))^{\gamma_{i_2}} \cdots (f^{(k)}(t))^{\gamma_{i_k}}, \quad t \in \mathbb{R}, \quad (2)$$

in which γ_{i_j} are the non-zero components of γ .

If $r, p \in \mathbb{Z}$ and $0 \leq r \leq p$, we define the set of multi-indexes

$$\Gamma_{r,p} = \left\{ \alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) : |\alpha| = \sum_{i=1}^n \alpha_i = r \text{ and } \sum_{i=1}^n i\alpha_i = p \right\}.$$

We now define the class of functions which will be crucial for our work.

Definition 2.1. *Given $s \geq 0$ and $\beta > 0$ we say that a real-valued function $a : \mathbb{R} \rightarrow \mathbb{R}$ of class C^∞ belongs to the class \mathcal{G}_s^β if it satisfies the following three conditions:*

G_1 . The function $t \mapsto a(t^2)$, $t \in \mathbb{R}$, is nonnegative.

G_2 . There exist positive constants R , M such that

$$M(1 + |x|^2)^{\frac{\beta}{2}} \leq a(|x|^2), \quad \text{for all } |x| > R, \quad x \in \mathbb{R}^n.$$

G_3 . For each $k \geq 0$ there exists a positive constant $N = N(k, s)$ such that

$$\left| \frac{d^k}{dt^k} a(t) \right|_{t=|x|^2} \leq N(1 + |x|^2)^{k(\frac{\beta s}{4n} - 1) + \frac{\beta}{2}}, \quad x \in \mathbb{R}^n. \quad (3)$$

Remark 2.1. We note that inequality (3) can be stated in a weaker form: the exponent appearing in the right hand side can be replaced by $kl + \frac{\beta}{2}$, in which l can be chosen arbitrarily, as long as it satisfies $l+1 \leq \beta s/4n$. We also note that Conditions G_1 – G_3 imply that the function $g : \mathbb{R}^n \rightarrow \mathbb{R}$, $g(x) = a(|x|^2)$, and its derivatives, are polynomially bounded. In Section 3 we will use the fact that this observation implies that if F is a tempered distribution on \mathbb{R}^n , so is gF , as noted in [25, p.137].

An interesting example of a function a belonging to the class \mathcal{G}_s^β is given by the following lemma:

Lemma 2.2. We take $0 < \alpha < 1$. Then, the function $f(t) = (t)^{\alpha/2}$ belongs to \mathcal{G}_s^α for all $s \geq 0$.

Proof. We first note that for $\alpha \geq 0$ we have the obvious estimate

$$|x|^\alpha \leq (1 + |x|^2)^{\alpha/2}. \quad (4)$$

Now, Condition G_1 obvious because $f(t^2) = (t^2)^{\alpha/2} = |t|^\alpha \geq 0$. In order to show that f satisfies G_2 , we note that $f(|x|^2) = |x|^\alpha$. Then, for $|x| \geq 1$ we have

$$(1 + |x|^2)^{\alpha/2} \leq (|x|^2 + |x|^2)^{\alpha/2} \leq 2^{\alpha/2} |x|^\alpha = 2^{\alpha/2} f(|x|^2),$$

and so f satisfies G_2 . Finally for G_3 , we note that for each $k > 0$ there exists a constant number C such that $f^{(k)}(t) = C t^{(\alpha-2k)/2}$. Thus, the elementary inequality (4) implies

$$|f^{(k)}(|x|^2)| = C |x|^{\alpha-2k} \leq C(1 + |x|^2)^{\alpha/2-k}.$$

Now, $k, \alpha, s \geq 0$ and so $\frac{k\alpha s}{4n} \geq 0$. It follows that

$$\left(\frac{\alpha}{2} - k\right) \leq \left(\frac{\alpha}{2} - k + \frac{k\alpha s}{4n}\right) = k\left(\frac{\alpha s}{4n} - 1\right) + \frac{\alpha}{2}.$$

Finally,

$$|f^{(k)}(|x|^2)| \leq C(1 + |x|^2)^{\alpha/2 - k} \leq C(1 + |x|^2)^{k(\frac{\alpha s}{4n} - 1) + \frac{\alpha}{2}}.$$

We conclude that f satisfies G_3 and so if $s \geq 0$ then the function $f(t)$ belongs to \mathcal{G}_s^α . \square

The following proposition is an easy consequence of the definitions. It allows us to order the classes \mathcal{G}_s^β .

Proposition 2.3. *Let us fix a number $\beta > 0$. If s_1 and s_2 are real numbers such that $0 \leq s_1 < s_2$, then $\mathcal{G}_{s_1}^\beta \subseteq \mathcal{G}_{s_2}^\beta$.*

Proof. Let us consider $a \in \mathcal{G}_{s_1}^\beta$. Conditions G_1 and G_2 depend only on β , so we only prove that G_3 holds if we use s_2 instead of s . We use the formulation appearing in the above remark; the result follows from the obvious facts that $s_1 < s_2$ implies $\frac{\beta s_1}{4n} < \frac{\beta s_2}{4n}$ and that $t \mapsto (1 + |x|^2)^t$ is increasing. \square

Lemma 2.4. *We fix an integer j and a multi-index α , and we take $\gamma \in \Gamma_{j,|\alpha|}$. Then, for any $a \in \mathcal{G}_s^\beta$, $\beta > 0$ and $s \geq 0$, there exists a constant C such that*

$$|a^\gamma(|x|^2)| \leq C(1 + |x|^2)^{|\alpha|l + \frac{j\beta}{2}}, \quad (5)$$

where $l + 1 \leq \frac{\beta s}{4n}$.

Proof. By definition given in (2) and since $a \in \mathcal{G}_s^\beta$ we have that

$$\begin{aligned} |a^\gamma(|x|^2)| &= \left| (a^{(1)}(|x|^2))^{\gamma_1} (a^{(2)}(|x|^2))^{\gamma_2} \dots (a^{(n)}(|x|^2))^{\gamma_n} \right| \\ &\leq C \left| (1 + |x|^2) \right|^{(l + \frac{\beta}{2})\gamma_1} \dots \left| (1 + |x|^2) \right|^{(nl + \frac{\beta}{2})\gamma_n} \\ &= C(1 + |x|^2)^{\sum_{i=1}^n (il + \frac{\beta}{2})\gamma_i} \\ &= C(1 + |x|^2)^{|\alpha|l + \frac{j\beta}{2}} \end{aligned}$$

\square

Now we let $a : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function and $s > 0$. We denote by $m_{a,s}$ the function

$$m_{a,s}(x) = \frac{1}{(1 + a(|x|^2))^{s/2}}, \quad x \in \mathbb{R}^n. \quad (6)$$

The following technical lemma will be important for the proof in Section 3, that $m_{a,s}$ is a Fourier multiplier.

Lemma 2.5. Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be a multi-index such that $\alpha_i \in \{0, 1\}$ for $i = 1, \dots, n$. Then

$$D^\alpha m_{a,s}(x) = x^\alpha \sum_{j=1}^{|\alpha|} \sum_{\gamma \in \Gamma_{j,|\alpha|}} C(\gamma, s) \frac{a^\gamma(|x|^2)}{(1 + a(|x|^2))^{\frac{s}{2}+j}}, \quad (7)$$

for some constants $C(\gamma, s)$.

Proof. We prove this lemma by induction on $k = |\alpha|$. First, let us note if $|\alpha| = 1$, then $\alpha = (0, \dots, \underbrace{1}_i, \dots, 0)$ i.e., $\alpha_k = 1$ if $k = i$ and $\alpha_k = 0$ if $k \neq j$. Therefore

$$D^\alpha m_{a,s}(x) = \frac{\partial}{\partial x_i} m_{a,s}(x) = -s \frac{x_i a'(|x|^2)}{(1 + a(|x|^2))^{\frac{s}{2}+1}},$$

and (7) holds with $C(\gamma, 1) = -s$, because $x_i = x^\alpha$ and the only multi-index γ belonging to $\Gamma_{1,1}$ is $(1, 0, \dots, 0)$.

Now we suppose that equality (7) is satisfied for all multi-indices α with $|\alpha| = k$; we consider a multi-index β , with $|\beta| = k + 1$. It is evident that if $|\beta| = k + 1$, then there exists some $i = 1, \dots, n$ and a multi-index α , with $|\alpha| = k$ such that

$$D^\beta = \frac{\partial}{\partial x_i} (D^\alpha),$$

and then

$$\begin{aligned} D^\beta m_{a,s}(x) &= \frac{\partial}{\partial x_i} (D^\alpha m_{a,s}(x)) \\ &= \frac{\partial}{\partial x_i} \left(x^\alpha \sum_{j=1}^k \sum_{\gamma \in \Gamma_{j,k}} C(\gamma, s) \frac{a^\gamma(|x|^2)}{(1 + a(|x|^2))^{\frac{s}{2}+j}} \right) \\ &= \left(x^\alpha \sum_{j=1}^k \sum_{\gamma \in \Gamma_{j,k}} C(\gamma, s) \frac{\partial}{\partial x_i} \frac{a^\gamma(|x|^2)}{(1 + a(|x|^2))^{\frac{s}{2}+j}} \right). \end{aligned} \quad (8)$$

Now, fixing $j \in \{1, \dots, k\}$ and $\gamma \in \Gamma_{j,k}$ and taking derivative of $\frac{a^\gamma(|x|^2)}{(1 + a(|x|^2))^{\frac{s}{2}+j}}$ with respect to x_i , we have for each summand of (8):

$$\begin{aligned} &\frac{\partial}{\partial x_i} [(a^\gamma(|x|^2))(1 + a(|x|^2))^{-s/2-j}] \\ &= \frac{\partial}{\partial x_i} [a^\gamma(|x|^2)](1 + a(|x|^2))^{-s/2-j} + a^\gamma(|x|^2) \frac{\partial}{\partial x_i} [(1 + a(|x|^2))^{-s/2-j}] \\ &= \frac{\partial}{\partial x_i} [a^\gamma(|x|^2)](1 + a(|x|^2))^{-s/2-j} + 2(-s/2 - j) \frac{x_i a^\gamma(|x|^2) a'(|x|^2)}{(1 + a(|x|^2))^{s/2+j+1}} \end{aligned} \quad (9)$$

To compute the first term of (9), we consider $\gamma = (\gamma_1, \dots, \gamma_n)$. Then,

$$\frac{\partial}{\partial x_i} a^\gamma(|x|^2) = x_i \sum_{r=1}^n \gamma_r a^{\bar{\gamma}^r}(|x|^2), \quad (10)$$

where the multi-index $\bar{\gamma}^r = (\bar{\gamma}_1^r, \bar{\gamma}_2^r, \dots, \bar{\gamma}_n^r)$, $r = 1, \dots, n$, is defined as follows:

$$\bar{\gamma}_l^r = \begin{cases} \gamma_r - 1 & \text{if } l = r \\ \gamma_r + 1 & \text{if } l = r + 1 \\ \gamma_r & \text{otherwise.} \end{cases}$$

Now we observe that since $\gamma \in \Gamma_{j,k}$, for each r we have

$$\begin{aligned} \sum_{l=1}^n \bar{\gamma}_l^r &= \bar{\gamma}_1^r + \dots + \bar{\gamma}_r^r + \bar{\gamma}_{r+1}^r + \dots + \bar{\gamma}_n^r \\ &= \gamma_1 + \dots + (\gamma_r - 1) + (\gamma_{r+1} + 1) + \dots + \gamma_n \\ &= \sum_{l=1}^n \gamma_l \\ &= j \end{aligned}$$

and

$$\begin{aligned} \sum_{l=1}^n l \bar{\gamma}_l^r &= \bar{\gamma}_1^r + 2\bar{\gamma}_2^r \dots + r\bar{\gamma}_r^r + (r+1)\bar{\gamma}_{r+1}^r + \dots + n\bar{\gamma}_n^r \\ &= \gamma_1 + 2\gamma_2 \dots + r(\gamma_r - 1) + (r+1)(\gamma_{r+1} + 1) + \dots + n\gamma_n \\ &= \left(\sum_{l=1}^n l \gamma_l \right) + 1 \\ &= k + 1. \end{aligned}$$

Hence, for each $r = 1 \dots, n$ we have that $\bar{\gamma}^r \in \Gamma_{j,(k+1)}$.

Now we compute the second term of (9). Let us observe that if $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n) \in \Gamma_{j,k}$, then

$$a^\gamma(|x|^2) a'(|x|^2) = a^{\gamma^*}(|x|^2),$$

where $\gamma^* = (\gamma_1^*, \gamma_2^*, \dots, \gamma_n^*)$ is such that $\gamma_1^* = \gamma_1 + 1$ and $\gamma_l^* = \gamma_l$ for all $2 \leq l \leq n$.

Hence, the second term of (9) is equal to

$$C(\gamma, s) \frac{x_i a^{\gamma^*}(|x|^2)}{(1 + a(|x|^2))^{s/2+j+1}}$$

where $\gamma^* \in \Gamma_{j+1,k+1}$. Thus, we have that for each $j \in \{1, \dots, k\}$ and $\gamma \in \Gamma_{j,k}$

$$\frac{\partial}{\partial x_i} \left(\frac{a^\gamma}{(1 + a(|x|^2))^{s/2+j}} \right) = x_i \left(\sum_{\bar{\gamma} \in \Gamma_{j,k+1}} C \frac{a^{\bar{\gamma}}(|x|^2)}{(1 + a(|x|^2))^{s/2+j}} + C \frac{a^{\gamma^*}(|x|^2)}{(1 + a(|x|^2))^{s/2+j+1}} \right)$$

in which $\gamma^* \in \Gamma_{j+1,k+1}$, and this fact proves the lemma. \square

Lemma 2.6. *Let us consider $m_{a,s}$ the function defined in (6). Then, for each positive integer k there are positive constants $C(k, s)$ such that*

$$\begin{aligned} \frac{\partial^k}{\partial x_i^k} m_{a,s}(x) &= \sum_{j=1}^k \sum_{\gamma \in \Gamma_{j,t}} C(k, s) \frac{a^\gamma(|x|^2) x_i^{(2t-k)}}{(1 + a(|x|^2))^{s/2+j}} \\ &= \sum_{j=1}^k \sum_{\gamma \in \Gamma_{j,t}} C(k, s) a^\gamma(|x|^2) x_i^{2t-k} m_{a,s+2j}(x), \end{aligned} \quad (11)$$

where $j \leq t \leq k$.

Proof. We shall prove this lemma by induction. First, let us observe that the lemma is obviously satisfied if $k = 1$ because

$$\frac{\partial}{\partial x_i} m_{a,s}(x) = -s a'(|x|^2) x_i m_{a,s+2}. \quad (12)$$

Now we assume that (11) is valid k ; we show that this identity is also verified for $(k+1)$:

$$\begin{aligned} \frac{\partial^{k+1}}{\partial x_i^{k+1}} m_{a,s}(x) &= \frac{\partial}{\partial x_i} \left(\frac{\partial^k}{\partial x_i^k} m_{a,s}(x) \right) \\ &= \sum_{j=1}^k \sum_{\gamma \in \Gamma_{j,t}} C \frac{\partial}{\partial x_i} (a^\gamma(|x|^2) x_i^{2t-k} m_{a,s+2j}) \end{aligned} \quad (13)$$

Now, if we fix $j \in \{1, \dots, k\}$ and $\gamma \in \Gamma_{j,t}$ for $j \leq t \leq k$, we observe that each term in (13) can be written as

$$\begin{aligned} \frac{\partial}{\partial x_i} (a^\gamma(|x|^2) x_i^{2t-k} m_{a,s+2j}) &= \frac{\partial}{\partial x_i} (a^\gamma(|x|^2)) x_i^{2t-k} m_{a,s+2j} \\ &+ a^\gamma(|x|^2) \frac{\partial}{\partial x_i} (x_i^{2t-k}) m_{a,s+2j} + a^\gamma(|x|^2) x_i^{2t-k} \frac{\partial}{\partial x_i} (m_{a,s+2j}) \end{aligned} \quad (14)$$

Since (10) holds, we have that if $\gamma \in \Gamma_{j,t}$, then $\frac{\partial}{\partial x_i} (a^\gamma(|x|^2)) = x_i \sum_{r=1}^n C a^{\bar{\gamma}_r}(|x|^2)$, where $\bar{\gamma}_r \in \Gamma_{j,(t+1)}$, thus, the first term of (14) is given by

$$\begin{aligned} \frac{\partial}{\partial x_i} (a^\gamma(|x|^2)) x_i^{2t-k} m_{a,s+2j} &= x_i \left(\sum_{r=1}^n C a^{\bar{\gamma}_r}(|x|^2) \right) x_i^{2t-k} m_{a,s+2j}(x) \\ &= \sum_{r=1}^n C a^{\bar{\gamma}_r}(|x|^2) x_i^{2(t+1)-(k+1)} m_{a,s+2j}(x) \end{aligned} \quad (15)$$

where for each $r = \{1, \dots, n\}$, the multi index $\overline{\gamma}_r \in \Gamma_{j,(t+1)}$.

Now, we compute the second term of (14). It is easy to see that

$$a^\gamma(|x|^2) \frac{\partial}{\partial x_i} (x_i^{2t-k}) m_{a,s+2j} = C a^\gamma(|x|^2) (x_i^{2t-(k+1)}) m_{a,s+2j}(x) \quad (16)$$

Finally, by Lemma 2.5 we can compute the third term of (14).

$$\begin{aligned} a^\gamma(|x|^2) x_i^{2t-k} \frac{\partial}{\partial x_i} (m_{a,s+2j}(x)) &= C a^\gamma(|x|^2) x_i^{2t-k} a'(|x|^2) x_i m_{a,s+2j+2} \\ &= C a^\zeta(|x|^2) x_i^{2(t+1)-(k+1)} m_{a,s+2j+2} \end{aligned} \quad (17)$$

where the multi index $\zeta \in \Gamma_{(j+1),(t+1)}$. Hence, replacing (15), (16) and (17) in (14) it is easy to see that the equality (11) is verified by the integer $(k+1)$, and thus the proof of the lemma follows. \square

3 Fourier multipliers and functional calculus

3.1 Fourier multipliers

Hereafter we write either $\mathcal{F}(f)$ or \widehat{f} for the Fourier transform of f .

Definition 3.1. Let m be a bounded measurable function on \mathbb{R}^n . We define a linear transformation T_m with domain $L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ as follows:

$$(T_m f)(x) = \mathcal{F}^{-1} \left(m(x) \widehat{f} \right), \quad f \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n). \quad (18)$$

We say that m is a Fourier multiplier for $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, if whenever $f \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ then $T_m f$ belongs to $L^p(\mathbb{R}^n)$ and T_m is bounded, that is,

$$\|T_m(f)\|_{L^p(\mathbb{R}^n)} \leq A \|f\|_{L^p(\mathbb{R}^n)}, \quad f \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n) \quad (19)$$

for a real constant A independent of f .

This definition is explained in [28, p. 94] and [2, p. 489]. Its importance for us is due to the following result, in which the relevance of our conditions G_1 – G_3 become clear.

Proposition 3.1. Let $\beta > 0$, $s_0 \geq 0$ and let $a \in \mathcal{G}_{s_0}^\beta$ be fixed. If $1 < p < \infty$, then the function $m_{a,s}$ defined in (6) is a Fourier multiplier for $L^p(\mathbb{R}^n)$ for all $s \geq s_0$.

Proof. Let $s \geq s_0$ be fixed. We show that for every multi-index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ such that $\alpha_i \in \{0, 1\}$, there exists a positive constant C such that for all $x \in \mathbb{R}^n$

$$|x^\alpha D^\alpha m_{a,s}(x)| \leq C \quad (20)$$

holds. It follows from Theorem IV.6' of [28], as explained in [20], that this is enough for proving the proposition.

Since for every $1 \leq i \leq n$ we have that $|x_i| \leq (1 + |x|^2)^{1/2}$. Thus it follows that for every multi index α

$$|x^\alpha| \leq (1 + |x|^2)^{\frac{|\alpha|}{2}}. \quad (21)$$

On the other hand, since $a \in \mathcal{G}_{s_0}^\beta$ and $s \geq s_0$, then Proposition 2.3 implies that $a \in \mathcal{G}_s^\beta$ as well, and therefore by the ellipticity condition G_2 , there exists M such that for all $|x| > R$ the inequality

$$\frac{1}{(1 + a(|x|^2))^{s/2}} \leq \frac{M}{(1 + |x|^2)^{(\beta s)/4}} \quad (22)$$

holds.

Now, according to Lemma 2.5 we have that

$$D^\alpha m_{a,s}(x) = x^\alpha \sum_{j=1}^{|\alpha|} \sum_{\gamma \in \Gamma_{j,|\alpha|}} C(\gamma, s) \frac{a^\gamma(|x|^2)}{(1 + a(|x|^2))^{\frac{s}{2}-j}}.$$

Since for every γ the function a^γ is continuous, then there exists a constant C_1 such that

$$|D^\alpha m_{a,s}(x)| \leq C_1, \quad \text{for } |x| \leq R.$$

But then, by (21) there exists constant C such that

$$|x^\alpha D^\alpha m_{a,s}(x)| \leq CR^n,$$

whenever $|x| \leq R$. Now we consider the case $|x| > R$. Since $a \in \mathcal{G}_{s_0}^\beta$, then by conditions G_2 and G_3 we have that for each multi index $\gamma \in \Gamma_{j,|\alpha|}$ there exist positive constants $C_1, C_2 \dots$ that we shall denote all by \tilde{C} , such that

$$\begin{aligned} & \left| x^\alpha \frac{a^\gamma(|x|^2)}{(1 + a(|x|^2))^{\frac{s+2j}{2}}} \right| \\ & \leq \tilde{C} |x|^{2|\alpha|} \left| \frac{(1 + |x|^2)^{|\alpha|(\frac{\beta s_0}{4n} - 1) + \frac{\beta j}{2}}}{(1 + |x|^2)^{\frac{\beta(s+2j)}{4}}} \right| \\ & \leq \tilde{C} (1 + |x|^2)^{|\alpha|} \left| \frac{(1 + |x|^2)^{|\alpha|(\frac{\beta s_0}{4n} - 1) + \frac{\beta j}{2}}}{(1 + |x|^2)^{\frac{\beta(s+2j)}{4}}} \right| \\ & = \tilde{C} (1 + |x|^2)^\mu \end{aligned}$$

in which

$$\begin{aligned}
\mu &= |\alpha| + |\alpha| \left(\frac{\beta s_0}{4n} - 1 \right) + \frac{\beta j}{2} - \frac{\beta s}{4} - \frac{\beta j}{2} \\
&= |\alpha| \left(\frac{\beta s_0}{4n} \right) - \frac{\beta s}{4} \\
&= \frac{\beta}{4} \left(\frac{|\alpha| s_0}{n} - s \right)
\end{aligned}$$

From the fact that $s_0 \leq s$, then $1 \leq \frac{s}{s_0}$. On the other hand, since $0 < \frac{|\alpha|}{n} < 1$, then

$$\frac{|\alpha|}{n} < \frac{s}{s_0},$$

and thus

$$\left(\frac{|\alpha| s_0}{n} - s \right) \leq 0,$$

and since $s_0, \beta > 0$, then we have that

$$\mu = \frac{\beta}{4} \left(\frac{|\alpha| s_0}{n} - s \right) \leq 0$$

i.e. for each multi index $\gamma \in \Gamma_{i,|\alpha|}$ we obtain the estimate

$$\left| x^\alpha \frac{a^\gamma(|x|^2)}{(1 + a(|x|^2))^{\left(\frac{s+2j}{2}\right)}} \right| \leq \tilde{C}$$

and hence, using Lemma 2.5, there exist a constant C such that

$$|x^\alpha D^\alpha m_{a,s}(x)| \leq C$$

for all $x \in \mathbb{R}^n$.

It follows from Theorem 2 in [20] that the function $m_{a,s}$ is a Fourier multiplier for $L^p(\mathbb{R}^n)$ for every $p \in (1, \infty)$ and for all $s \geq s_0$. \square

Now we consider the Fourier multiplier $m_{a,s}$ and the induced linear transformation $T_{m_{a,s}}$ introduced in Definition 3.1. An important characteristic of $T_{m_{a,s}}$ is the following invariance property:

Proposition 3.2. *We set $T_s = T_{m_{a,s}}$ on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, that is,*

$$T_s(u) = \mathcal{F}^{-1} \left(\frac{1}{(1 + a(|x|^2))^{s/2}} \widehat{u} \right). \quad (23)$$

Then, T_s is a bounded translation-invariant operator from $L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.

Proof. Since $m_{a,s}$ is a Fourier multiplier, we know that T_s is a bounded linear operator from $L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. We write

$$T_s(u) = \mathcal{F}^{-1}(m_{a,s}(x) \widehat{u}).$$

Then, invariance follows from well-known properties of Fourier transform, and the fact that $m_{a,s}$ is rotationally invariant. \square

We end this subsection by observing, after [28, p. 94], that if $p < \infty$ then Definition 3.1 implies that T_m has a unique bounded extension to $L^p(\mathbb{R}^n)$. Following standard usage, we keep writing T_m for this extension.

3.2 Functional calculus for \mathcal{G}_s^β symbols

We show in this subsection that symbols in \mathcal{G}_s^β define suitable operators on $L^p(\mathbb{R}^n)$. Interestingly, these operators can be characterized as convolution operators with a kernel K which we now introduce.

Lemma 3.3. *Let $\beta > 0, s \geq 0$ and $a \in \mathcal{G}_s^\beta$. Let us consider*

$$K(x) = \int_{\mathbb{R}^n} \frac{e^{i\xi x}}{(1 + a(|\xi|^2))^{s/2}} d\xi. \quad (24)$$

Then, $K \in L^1(\mathbb{R}^n)$ whenever β, s are such that $\beta s > 2n$.

Proof. First, let us note that the ellipticity condition (G_2) yields the inequality

$$\frac{1}{(1 + a(|x|^2))^{s/2}} \leq \frac{M}{(1 + |x|^2)^{\frac{\beta s}{4}}} \quad (25)$$

for $|x| > R$. Thus we have that for all $x \in \mathbb{R}^n$ there exists a positive constant M such that

$$\begin{aligned} |K(x)| &\leq \int_{\mathbb{R}^n} \frac{dx}{(1 + a(|\xi|^2))^{s/2}} \\ &= \int_{|\xi| \leq R} \frac{dx}{(1 + a(|\xi|^2))^{s/2}} + \int_{|\xi| > R} \frac{dx}{(1 + a(|\xi|^2))^{s/2}} \\ &\leq \int_{|\xi| \leq R} \frac{dx}{(1 + a(|\xi|^2))^{s/2}} + \int_{|\xi| > R} \frac{M dx}{(1 + a(|\xi|^2))^{\beta s/4}}. \end{aligned} \quad (26)$$

The first of the integrals in (26) is obviously finite, while the second one is finite as well because of ellipticity (condition G_2 of Definition 2.1) and our assumption

$\beta s > 2n$. Hence, there exists a constant $C > 0$ such that $|K(x)| \leq C$ for all $x \in \mathbb{R}^n$. Now let us consider $k > n$; integrating by parts we have

$$\begin{aligned} x_i^k K(x) &= \int_{\mathbb{R}^n} \frac{(x_i)^k e^{ix\xi}}{(1 + a(|\xi|^2))^{s/2}} d\xi \\ &= \frac{1}{i^k} \int_{\mathbb{R}^n} e^{ix\xi} \frac{\partial^k}{\partial \xi_i^k} \left(\frac{1}{(1 + a(|\xi|^2))^{s/2}} \right) d\xi, \end{aligned}$$

therefore, by Lemma 2.6 we have,

$$\begin{aligned} |(x_i)^k K(x)| &\leq \int_{\mathbb{R}^n} \left| \frac{\partial^k}{\partial \xi_i^k} \left(\frac{1}{(1 + a(|\xi|^2))^{s/2}} \right) \right| d\xi \\ &\leq \int_{\mathbb{R}^n} \left| \sum_{m=1}^k \sum_{\gamma \in \Gamma_{m,p}} C(m, p) \frac{a^\gamma(|\xi|^2) \xi_i^{2p-j}}{(1 + a(|\xi|^2))^{\frac{s}{2}+m}} \right| d\xi. \\ &\leq \int_{|\xi| \leq R} \left| \sum_{m=1}^k \sum_{\gamma \in \Gamma_{m,p}} C(m, p) \frac{a^\gamma(|\xi|^2) \xi_i^{2p-j}}{(1 + a(|\xi|^2))^{\frac{s}{2}+m}} \right| d\xi \\ &\quad + \int_{|\xi| > R} \left| \sum_{m=1}^k \sum_{\gamma \in \Gamma_{m,p}} C(m, p) \frac{a^\gamma(|\xi|^2) \xi_i^{2p-j}}{(1 + a(|\xi|^2))^{\frac{s}{2}+m}} \right| d\xi. \end{aligned}$$

Observe now that for all $m = 1, \dots, k$ and $\gamma \in \Gamma_{m,p}$ there exists a positive constant C such that

$$\begin{aligned} \left| \frac{a^\gamma(|\xi|^2) \xi_i^{2p-k}}{(1 + a(|\xi|^2))^{\frac{s}{2}+m}} \right| &\leq \frac{C(1 + |\xi|^2)^{p(\frac{\beta s}{4n}-1) + \frac{m\beta}{2}} (1 + |\xi|^2)^{p-\frac{k}{2}}}{(1 + |\xi|^2)^{\frac{\beta}{4}(s+2m)}} \\ &= \frac{C(1 + |\xi|^2)^{p(\frac{\beta s}{4n}) - \frac{k}{2}}}{(1 + |\xi|^2)^{\frac{\beta s}{4}}} \\ &= \frac{C}{(1 + |\xi|^2)^{\frac{\mu}{2}}} \end{aligned}$$

where $\mu = \frac{\beta s}{2n}(n - p) + k > n - p + k \geq n$.

Hence,

$$|(x_i)^k K(x)| \leq \int_{\mathbb{R}^n} \frac{C}{(1 + |\xi|^2)^{\mu/2}} d\xi \leq C\pi^n \quad (27)$$

On other hand, for any $r > 0$ there exists a positive $C(r) = C$ such that

$$(1 + |x|^2)^{r/2} \leq C(1 + |x|^r).$$

Hence there exists $C > 0$ so that

$$|x|^r \leq C(x_1^r + \dots + x_n^r).$$

Therefore, applying (27) and choosing an appropriated $r > 0$, we obtain that

$$\begin{aligned} |(1 + |x|^2)^{r/2} K(x)| &\leq C|(1 + x_1^r + \dots + x_n^r) K(x)| \\ &\leq C(n\pi^n) . \end{aligned} \tag{28}$$

This inequality implies that $K \in L^1(\mathbb{R}^n)$. \square

We now define an operator A on $L^p(\mathbb{R}^n)$ for each symbol $a \in \mathcal{G}_s^\beta$. We assume without further mention that each time we consider Fourier transform (or inverse Fourier transform) of elements in $L^p(\mathbb{R}^n)$, we are identifying these elements with appropriate tempered distributions.

Definition 3.2. Let $a \in \mathcal{G}_s^\beta$, $\beta > 0$ and $s \geq 0$. We set

$$A(u) := \mathcal{F}^{-1} \left((1 + a(|\xi|^2))^{s/2} \widehat{u} \right)$$

on $L^p(\mathbb{R}^n)$ with domain

$$D(A) = \{u \in L^p(\mathbb{R}^n) : \mathcal{F}^{-1}((1 + a(|\xi|^2))^{s/2} \widehat{u}) \in L^p(\mathbb{R}^n)\}.$$

Remark 3.4. We need to check that $D(A)$ is non-trivial. Indeed, let $a \in \mathcal{G}_s^\beta$, $\beta > 0$, $s \geq 0$. We consider the bounded operator T_s defined in Proposition 3.2. It is easy to see, using the fact that Fourier transform is an isomorphism at the level of tempered distributions, that $u \in L^p(\mathbb{R}^n)$ belongs to $D(A)$ if and only if

$$u = T_s(g) \tag{29}$$

for some $g \in L^p(\mathbb{R}^n)$.

The next proposition is an application of Theorem 1.2 in Hörmander's paper [21]:

Proposition 3.5. Let $a \in \mathcal{G}_s^\beta \cap C^\infty(\mathbb{R})$, $\beta > 0$, $s \geq 0$ and assume that $\beta s > 2n$. Then, a function u belongs to $D(A) \cap \mathcal{S}(\mathbb{R}^n)$ if and only if $u = K * g$ for $g \in \mathcal{S}(\mathbb{R}^n)$, in which K has been defined in (24).

Proof. By the remark above, if u belongs to $D(A) \cap \mathcal{S}(\mathbb{R}^n)$, we can write u as in (29) for some $g \in L^p(\mathbb{R}^n)$. Equation (29) then implies that

$$(1 + a(|\xi|^2))^{s/2} \widehat{u} = \widehat{g} .$$

Since u belongs to $D(A) \cap \mathcal{S}(\mathbb{R}^n)$ and the function $\xi \mapsto a(|\xi|^2)$ is polynomially bounded, the left hand side of the above equation belongs to Schwarz space, and therefore so does g .

Now, by Proposition 3.1, the function $M(\xi) = \frac{1}{(1 + a(|\xi|^2))^{s/2}}$ is a Fourier multiplier for $L^p(\mathbb{R}^n)$ and therefore (see Proposition 3.2 and remark below it) the unique continuous extension of the map

$$g \mapsto T_s(g) = \mathcal{F}^{-1} \left(\frac{1}{(1 + a(|\xi|^2))^{s/2}} \widehat{g}(\xi) \right) \quad (30)$$

is a translation invariant bounded operator from $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$. In particular, if $g \in \mathcal{S}(\mathbb{R}^n)$, we compute $T_s(g)$ using exactly (30). By Theorem 1.2 in Hörmander's paper [21], we conclude that there exists a unique distribution $\Theta \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$T_s(g) = \Theta * g \quad (31)$$

for all $g \in \mathcal{S}(\mathbb{R}^n)$. We claim that Θ is exactly K as defined in (24). Indeed,

$$\mathcal{F}(T_s(g)) = M \widehat{g} = \widehat{\widehat{M}} \widehat{g} = \mathcal{F}(\widehat{M} * g) .$$

Thus, $T_s(g) = \widehat{M} * g$ for $g \in \mathcal{S}(\mathbb{R}^n)$. By uniqueness of the distribution Θ , we have $\Theta = \widehat{M} = K$.

Conversely, if $u = K * g$ for $g \in \mathcal{S}(\mathbb{R}^n)$, then $\widehat{u} = \widehat{K} \widehat{g}$. It follows that $u \in \text{Dom}(A)$. Moreover, we can easily see that $K \in \mathcal{S}(\mathbb{R}^n)$ since condition G_3 is satisfied, and therefore we conclude that $u \in \mathcal{S}(\mathbb{R}^n)$. \square

We are now ready to make the following crucial definition:

Definition 3.3. *Given $s \geq 0$, $\beta > 0$ and a in the class \mathcal{G}_s^β , we define the space $\mathcal{H}^{s,p}(a)$ as follows:*

$$\mathcal{H}^{s,p}(a) = \left\{ u \in L^p(\mathbb{R}^n) : \mathcal{F}^{-1} \left((1 + a(|\cdot|^2))^{s/2} \mathcal{F}(u) \right) \in L^p(\mathbb{R}^n) \right\} .$$

Thus, if $s \geq 0$, $\beta > 0$ and $a \in \mathcal{G}_s^\beta$ then, $D(A) = \mathcal{H}^{s,p}(a)$.

Remark 3.6. *The $L^2(\mathbb{R}^n)$ case of Definition 3.3 was treated in [17]. Interestingly, that paper was motivated by our previous work on the non-linear equation*

$$\Delta \exp(-c\Delta)u = U(\cdot, u) ,$$

see [19, 18]. In the context of the present paper is natural to consider the function $a(t) = t \exp(ct)$ as the corresponding symbol associated to the operator $-\Delta \exp(-c\Delta)$, and to study the equation $\Delta \exp(-c\Delta)u = -U(\cdot, u)$ instead of the equation above. It is not difficult to realize that the function

$$m_{\exp}(x) = \frac{1}{1 + |x|^2 e^{c|x|^2}} \quad (32)$$

defines a Fourier multiplier for $L^p(\mathbb{R}^n)$, even though it does not belongs to any of the spaces of symbols \mathcal{G}_s^β . It follows that we can also use this symbol a in Definition 3.3, thereby yielding a $L^p(\mathbb{R}^n)$ -version of the theory developed in [17, 19, 18].

We endow the space $\mathcal{H}^{s,p}(a)$ with the norm $\|\cdot\|_{\mathcal{H}^{s,p}(a)}$ defined in (33) below:

Theorem 3.7. *Let $s \geq 0$, $\beta > 0$ and $a \in \mathcal{G}_s^\beta$. The space $\mathcal{H}^{s,p}(a)$ endowed with the norm*

$$\|u\|_{\mathcal{H}^{s,p}(a)} = \|\mathcal{F}^{-1}((1 + a(|\cdot|^2))^{s/2} \mathcal{F}(u))\|_{L^p(\mathbb{R}^n)}. \quad (33)$$

is a Banach space.

The proof is immediate since the operator T_s (see Proposition 3.2 and Remark 3.4) and its inverse $u \mapsto \mathcal{F}^{-1}((1 + a(|\cdot|^2))^{s/2} \mathcal{F}(u))$ are continuous, as discussed in the proof of Proposition 3.5. Since we can write $\|u\|_{\mathcal{H}^{s,p}(a)} = \|T_s^{-1}(u)\|_{L^p(\mathbb{R}^n)}$, it is standard to conclude that $\mathcal{H}^{s,p}(a)$ is indeed a Banach space.

The Banach space $\mathcal{H}^{s,p}(a)$ embeds continuously into $L^p(\mathbb{R}^n)$. Indeed:

Theorem 3.8. *Let $s \geq 0$, $\beta > 0$ and $a \in \mathcal{G}_s^\beta$. Then the inclusion map $\mathcal{H}^{s,p}(a) \hookrightarrow L^p(\mathbb{R}^n)$ is continuous.*

Proof. Observe that for all $u \in L^p(\mathbb{R}^n)$

$$\begin{aligned} \|u\|_{L^p(\mathbb{R}^n)} &= \left\| \mathcal{F}^{-1} \left(\frac{1}{(1 + a(|\xi|^2))^{s/2}} (1 + a(|\xi|^2))^{s/2} \mathcal{F}(u) \right) \right\|_{L^p(\mathbb{R}^n)} \\ &= \|T_s(\mathcal{F}^{-1}((1 + a(|\xi|^2))^{s/2} \mathcal{F}(u)))\|_{L^p(\mathbb{R}^n)}. \end{aligned} \quad (34)$$

Since T_s is a bounded operator on $L^p(\mathbb{R}^n)$, there exists a positive constant C such that

$$\begin{aligned} \|T_s(\mathcal{F}^{-1}((1 + a(|\xi|^2))^{s/2} \mathcal{F}(u)))\|_{L^p(\mathbb{R}^n)} &\leq C \|\mathcal{F}^{-1}((1 + a(|\xi|^2))^{s/2} \mathcal{F}(u))\| \\ &= C \|u\|_{\mathcal{H}^{s,p}(a)}. \end{aligned}$$

Hence, there exists positive constant C such that $\|u\|_{L^p(\mathbb{R}^n)} \leq C \|u\|_{\mathcal{H}^{s,p}(a)}$. \square

We now present two further embedding results for the spaces $\mathcal{H}^{s,p}(a)$. We need two preliminary lemmas.

Lemma 3.9. *Let $\beta > 0$, $s \geq 0$ and $a \in \mathcal{G}_s^\beta$. If $r > 0$, then the function φ defined by*

$$\varphi(x) = \frac{(1 + |x|^2)^{r/2}}{(1 + a(|x|^2))^{\frac{1}{2}(s + \frac{2r}{\beta})}} \quad (35)$$

is a Fourier multiplier for $L^p(\mathbb{R}^n)$.

Proof. It is clear that the function φ can be written as

$$\varphi(x) = m_{a,s+\frac{2r}{\beta}}(x) (1 + |x|^2)^{r/2} \quad (36)$$

where $m_{a,s}$ is the function defined in (6), thus, for each multi index $\alpha \leq (1, \dots, 1)$

$$x^\alpha D^\alpha (\varphi(x)) = x^\alpha D^\alpha \left(m_{a,s+\frac{2r}{\beta}}(x) (1 + |x|^2)^{r/2} \right)$$

and by Leibniz's formula (see Theorem 1.2 in [26]) it follows that

$$x^\alpha D^\alpha (\varphi(x)) = x^\alpha \sum_{\gamma} \binom{\alpha}{\gamma} D^\gamma \left(m_{a,s+\frac{2r}{\beta}}(x) \right) D^{(\alpha-\gamma)} \left((1 + |x|^2)^{r/2} \right),$$

where $\gamma \leq \alpha$. Then, since

$$D^{(\alpha-\gamma)} \left((1 + |x|^2)^{r/2} \right) = C(r) x^{(\alpha-\gamma)} (1 + |x|^2)^{\frac{r}{2} - (|\alpha| - |\gamma|)}$$

and by Lemma (2.5)

$$D^\gamma \left(m_{a,s+\frac{2r}{\beta}}(x) \right) = x^\gamma \sum_{j=1}^{|\gamma|} \sum_{\mu \in \Gamma_{j,|\gamma|}} C(\mu, s) \frac{a^\mu(|x|^2)}{(1 + a(|x|^2))^{\frac{1}{2}(s+\frac{2r}{\beta})+j}},$$

we have

$$x^\alpha D^\alpha (\varphi(x)) = \sum_{\gamma} \binom{\alpha}{\gamma} \sum_{j=1}^{|\gamma|} \sum_{\mu \in \Gamma_{j,|\gamma|}} C(r, \mu, s) \frac{x^{2\alpha} (1 + |x|^2)^{\frac{r}{2} - |\alpha| + |\gamma|} a^\mu(|x|^2)}{(1 + a(|x|^2))^{\frac{1}{2}(s+\frac{2r}{\beta})+2j}}$$

and since

$$\begin{aligned} \left| \frac{x^{2\alpha} (1 + |x|^2)^{\frac{r}{2} - |\alpha| + |\gamma|} a^\mu(|x|^2)}{(1 + a(|x|^2))^{\frac{1}{2}(s+\frac{2r}{\beta})+2j}} \right| &\leq C \frac{(1 + |x|^2)^{|\alpha|} (1 + |x|^2)^{\frac{r}{2} - |\alpha| + |\gamma|} (1 + |x|^2)^{|\gamma|(\frac{\beta s}{4n} - 1) + \frac{j\beta}{2}}}{(1 + |x|^2)^{\frac{\beta}{4}(s+\frac{2r}{\beta})+2j}} \\ &= C (1 + |x|^2)^{|\gamma|(\frac{\beta s}{4n}) - \frac{\beta s}{4}} \\ &= C (1 + |x|^2)^{\frac{\beta s}{4}(\frac{|\gamma|}{n} - 1)} \end{aligned}$$

and since $|\gamma| \leq |\alpha| \leq n$, then we have $C(1 + |x|^2)^{\frac{\beta s}{4}(\frac{|\gamma|}{n} - 1)} \leq C$ and therefore for all multi index $\alpha \leq (1, \dots, 1)$

$$|x^\alpha D^\alpha (\varphi(x))| \leq C,$$

hence the function φ is a Fourier multiplier for $L^p(\mathbb{R}^n)$. \square

Lemma 3.10. *Let us consider the operator Λ on $L^p(\mathbb{R}^n)$ defined by*

$$\Lambda(u) = \mathcal{F}^{-1}(\varphi(x)\mathcal{F}(u)), \quad (37)$$

where φ is the function defined in (35). Then, there exists a positive constant C such that

$$\|\Lambda(u)\|_{L^p(\mathbb{R}^n)} \leq C\|u\|_{L^p(\mathbb{R}^n)}. \quad (38)$$

This Lemma follows from the fact that φ is a Fourier multiplier for $L^p(\mathbb{R}^n)$. Our promised embedding results for the spaces $\mathcal{H}^{s,p}(a)$ are:

Theorem 3.11. *Let $\beta > 0, s \geq 0$ and $a \in \mathcal{G}_s^\beta$ be fixed. Then, for each $r \geq 0$ the following continuous embedding*

$$\mathcal{H}^{s+\frac{2r}{\beta},p}(a) \hookrightarrow H^{r,p}(\mathbb{R}^n), \quad (39)$$

holds, in which $H^{r,p}(\mathbb{R}^n)$ is classical fractional Sobolev space as defined for example in [31].

Proof. Let us set $s_0 = s + \frac{2r}{\beta}$. Then, Proposition 2.3 implies that $a \in \mathcal{G}_{s_0}^\beta$. We now observe that for $u \in \mathcal{H}^{s_0,p}(a)$ we have

$$\begin{aligned} \|u\|_{H^{r,p}(\mathbb{R}^n)} &= \|\mathcal{F}^{-1}((1+|x|^2)^{r/2}\mathcal{F}(u))\|_{L^p(\mathbb{R}^n)} \\ &= \left\| \mathcal{F}^{-1} \left(\frac{(1+|x|^2)^{r/2}}{(1+a(|x|^2))^{s_0/2}} (1+a(|x|^2))^{s_0/2} \mathcal{F}(u) \right) \right\|_{L^p(\mathbb{R}^n)} \\ &= \|\Lambda [\mathcal{F}^{-1}((1+a(|x|^2))^{s_0/2}\mathcal{F}(u))]\|_{L^p(\mathbb{R}^n)} \\ &\leq C\|\mathcal{F}^{-1}((1+a(|x|^2))^{s_0/2}\mathcal{F}(u))\|_{L^p(\mathbb{R}^n)} \\ &= C\|u\|_{\mathcal{H}^{s_0,p}(a)}, \end{aligned}$$

in which we have used that $\mathcal{F}^{-1}((1+a(|x|^2))^{s_0/2}\mathcal{F}(u)) \in L^p(\mathbb{R}^n)$ since $u \in \mathcal{H}^{s_0,p}(a)$. \square

Corollary 3.12. *Let $\beta > 0, s \geq 0$ and $a \in \mathcal{G}_s^\beta$ be fixed. If $u \in \mathcal{H}^{s+\frac{2r}{\beta},p}(a)$ and $r > n/p$, then the function u is continuous and bounded.*

Proof. The corollary follows from the above theorem and Prop. 6.3 of [31]. \square

Theorem 3.13. *Let $\beta > 0, s \geq 0$ and $a \in \mathcal{G}_s^\beta$ be fixed. For each $\delta \geq 0$ the following continuous embedding*

$$\mathcal{H}^{2s+\delta,p}(a) \hookrightarrow \mathcal{H}^{s,p}(a) \quad (40)$$

holds.

Proof. As in the previous theorem, Proposition 2.3 implies that $a \in \mathcal{G}_{2s+\delta}^\beta$. Now we observe that for all $u \in \mathcal{H}^{2s+\delta,p}(a)$ we have

$$\begin{aligned}
\|u\|_{\mathcal{H}^{s,p}(a)} &= \left\| \mathcal{F}^{-1} \left(\frac{(1 + a(|x|^2))^{s/2}}{(1 + a(|x|^2))^{\frac{1}{2}(2s+\delta)}} (1 + a(|x|^2))^{\frac{1}{2}(2s+\delta)} \mathcal{F}(u) \right) \right\|_{L^p(\mathbb{R}^n)} \\
&= \left\| \mathcal{F}^{-1} \left(\frac{1}{(1 + a(|x|^2))^{(s+\delta)/2}} (1 + a(|x|^2))^{\frac{1}{2}(2s+\delta)} \mathcal{F}(u) \right) \right\|_{L^p(\mathbb{R}^n)} \\
&= \left\| T_{s+\delta} \left(\mathcal{F}^{-1} \left[(1 + a(|x|^2))^{\frac{1}{2}(2s+\delta)} \mathcal{F}(u) \right] \right) \right\|_{L^p(\mathbb{R}^n)} \\
&\leq C \left\| \mathcal{F}^{-1} \left[(1 + a(|x|^2))^{\frac{1}{2}(2s+\delta)} \mathcal{F}(u) \right] \right\|_{L^p(\mathbb{R}^n)} \\
&= C \|u\|_{\mathcal{H}^{2s+\delta,p}(a)} .
\end{aligned}$$

□

4 Nonlinear Equations associated to $a(-\Delta)$

The main aim of this section is the study in the spaces $L^p(\mathbb{R}^n)$ of the equation

$$(a(-\Delta) + 1)^{s/2} u = V(x, u) \quad (41)$$

in which $s \geq 0$, and the non-linearity V satisfies some technical hypotheses to be specified below.

As already discussed in Section 1, this equation encompasses several special cases of interest. If $s = 2$, Equation (41) becomes

$$(a(-\Delta) + 1) u = V(u)$$

so that, setting $a(-\Delta) + 1 = f(-\Delta)$, we arrive at Equation (13) of [17]. Also, Equation (41) generalizes the non-linear Laplace equation in two ways: considering $a(-\Delta) + 1 = (-\Delta)^\alpha$ (which is allowed because of Lemma 2.2) and $s = 2$, we obtain the non-linear fractional Laplace equation

$$(-\Delta)^\alpha u = V(x, u) , \quad (42)$$

while setting $a(-\Delta) + 1 = f(-\Delta)$ yields the equation

$$f(-\Delta)^{s/2} u = V(x, u) , \quad (43)$$

which can be thought of as generalizing $f(-\Delta) u = V(x, u)$ in a similar way as $(-\Delta)^\alpha u = V(x, u)$ generalizes the classical non-linear Laplace equation $-\Delta u = V(x, u)$.

First of all we solve the linear problem

$$(a(-\Delta) + 1)^{s/2} u = g \quad (44)$$

on $\mathcal{H}^{s,p}(a)$:

Theorem 4.1. *Let $\beta > 0, s \geq 0$ and $a \in \mathcal{G}_s^\beta$. For each $g \in L^p(\mathbb{R}^n)$ there exists an unique solution $u_g \in \mathcal{H}^{s,p}(a)$ to the linear equation (44). Moreover, we have*

$$\|u_g\|_{\mathcal{H}^{s,p}(a)} = \|g\|_{L^p(\mathbb{R}^n)} . \quad (45)$$

Proof. Equation (44) is equivalent to

$$\mathcal{F}^{-1} \left((1 + a(|\xi|^2))^{s/2} \mathcal{F}(u) \right) = g ,$$

and therefore it is easy to see that the solution u_g is given by

$$u_g = \mathcal{F}^{-1} \left(\frac{\mathcal{F}(g)}{(1 + a(|\xi|^2))^{s/2}} \right) .$$

This solution belongs to $\mathcal{H}^{s,p}(a)$ by Corollary 3.5. \square

We are ready to study non-linear equations. First of all, we state the following elementary proposition, showing that in some cases we can obtain existence, *uniqueness* and regularity of solutions.

Proposition 4.2. *Let us assume that $1 < p < \infty$ and that $p^{-1} + q^{-1} = 1$. Let $\beta > 2n/p, s \geq 0$ and $a \in \mathcal{G}_{s+1}^\beta$ be fixed. Suppose further that the function $V : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{C}$ is such that $V(\cdot, 0) \in L^p(\mathbb{R}^n)$, and that there exists a function $h \in L^{pq}(\mathbb{R}^n)$ such that*

$$|V(x, y_1) - V(x, y_2)| \leq h(x) |y_1 - y_2|^{1/p} .$$

Then, for all $\delta > 0$ small enough, the equation

$$[1 + a(-\Delta)]^{s/2} (u) = \delta V(\cdot, u) \quad (46)$$

has a unique solution $u \in \mathcal{H}^{s+1,p}(a)$. By Theorem 3.8, the solution is in $L^p(\mathbb{R}^n)$, and moreover, this solution is bounded and continuous.

Proof. Existence and uniqueness follow from the Banach Fixed Point Theorem. Regularity follows because if $r = \beta/2$, then Theorem 3.11 implies that the solution u belongs to the Sobolev space $H^{\beta/2,p}(\mathbb{R}^n)$, and therefore Prop. 6.3 of [31] tells us that u is bounded and continuous. \square

The next theorem establishes existence and regularity of solutions in a less restrictive framework.

Theorem 4.3. *Let $\beta > 0, s \geq 0$, and $a \in \mathcal{G}_s^\beta$ be given, and let $1 < p < \infty$. Given $\delta > 0$, we consider the equation*

$$[1 + a(-\Delta)]^{s/2}(u) = \delta\varphi(x)V(x, u) \quad (47)$$

in which $\varphi \in C_0^\infty$ and $V \in C^1(\mathbb{R}^n \times \mathbb{R})$. Let us assume that there exist constant numbers $\alpha > 1$ and $C > 0$, and a function $h \in L^p(\mathbb{R}^n)$ such that the function V satisfies the following estimates

$$|V(x, y)| + \left| \frac{\partial}{\partial x_i} V(x, y) \right| \leq C(h(x) + |y|^\alpha) \quad i = 1, 2, \dots, n \quad (48)$$

$$\left| \frac{\partial}{\partial y} V(x, y) \right| \leq C(1 + |y|^\alpha). \quad (49)$$

If we take $m > 0$ and the parameters β, s, m satisfy $m > n/(\alpha p)$ and $s > 4m\alpha/\beta$, then, for δ sufficiently small, Equation (47) has a solution $u \in \mathcal{H}^{s,p}(a)$.

Proof. Let us set $r_\alpha = \frac{n(\alpha - 1)}{p\alpha}$. We claim that if $u \in H^{r_\alpha, p}(\mathbb{R}^n)$ is given, then the function $V(\cdot, u) \in L^p(\mathbb{R}^n)$. In fact, since $r_\alpha p < n$, the continuous inclusion

$$H^{r_\alpha, p}(\mathbb{R}^n) \hookrightarrow L^{\alpha p}(\mathbb{R}^n) \quad (50)$$

holds (see for instance Prop. 6.4, chapter 13 of [31]), and therefore

$$\|u\|_{L^{\alpha p}(\mathbb{R}^n)} \leq C\|u\|_{H^{r_\alpha, p}(\mathbb{R}^n)}$$

for all $u \in H^{r_\alpha, p}(\mathbb{R}^n)$. An easy calculation using (48) now yields

$$\begin{aligned} \|V(\cdot, u)\|_{L^p(\mathbb{R}^n)}^p &\leq C \left(\|h\|_{L^p(\mathbb{R}^n)}^p + \int_{\mathbb{R}^n} |u(x)^\alpha|^p dx \right) \\ &= C \left(\|h\|_{L^p(\mathbb{R}^n)}^p + \|u\|_{L^{\alpha p}}^{\alpha p} \right) \end{aligned} \quad (51)$$

and the claim follows. Since $\varphi \in L^\infty(\mathbb{R}^n)$, we also conclude that the function $\varphi V(\cdot, u)$ belongs to $L^p(\mathbb{R}^n)$.

Now we need to use the following chain of continuous inclusions: we have (because of Theorem 3.13, Theorem 3.11 and [23])

$$\mathcal{H}^{s,p}(a) \hookrightarrow \mathcal{H}^{s/2-\epsilon,p}(a) = \mathcal{H}^{2(r_\alpha+m)/\beta,p}(a) \hookrightarrow H^{r_\alpha+m,p}(\mathbb{R}^n) \hookrightarrow H^{r_\alpha,p}(\mathbb{R}^n), \quad (52)$$

in which m is as in the enunciate of the theorem and ϵ is determined by the equation $s/2 - \epsilon = 2(r_\alpha + m)/\beta$. By the hypotheses of the theorem, we obtain $\epsilon > 0$.

Next, we set $A_0 = \{u \in H^{r_\alpha+m,p}(\mathbb{R}^n) : \|u\|_{H^{r_\alpha+m,p}(\mathbb{R}^n)} \leq 1\}$ and we define the operator $\mathcal{R} : A_0 \rightarrow A_0$ as follows:

$$\mathcal{R}(u) = w$$

in which w is the unique solution to the linear equation

$$[1 + a(-\Delta)]^{s/2} w = \delta \varphi V(\cdot, u) . \quad (53)$$

Since $\delta \varphi V(\cdot, u) \in L^p(\mathbb{R}^n)$, Theorem 4.1 implies that there exists an unique solution w to the equation (53) and therefore \mathcal{R} is well-defined. We now check that its range is indeed A_0 if we choose δ appropriately.

Since a belongs to \mathcal{G}_s^β , Theorem 4.1 tells us that the solution $w = \mathcal{R}(u)$ belongs to $\mathcal{H}^{s,p}(a)$. Inclusions (52) imply that w belongs to $H^{r_\alpha+m,p}(\mathbb{R}^n)$. We have that

$$\|\mathcal{R}(u)\|_{H^{r_\alpha+m,p}(\mathbb{R}^n)} = \|w\|_{H^{r_\alpha+m,p}(\mathbb{R}^n)} ,$$

and then the following inequalities hold:

$$\begin{aligned} \|w\|_{H^{r_\alpha+m,p}(\mathbb{R}^n)} &\leq C \|w\|_{\mathcal{H}^{s,p}(a)} \\ &= C \|\delta \varphi V(\cdot, u)\|_{L^p(\mathbb{R}^n)} \\ &\leq C \delta \|\varphi\|_{L^\infty(\mathbb{R}^n)} \left(\|h\|_{L^p(\mathbb{R}^n)}^p + \tilde{C} \|u\|_{H^{r_\alpha+m,p}(\mathbb{R}^n)}^{\alpha p} \right) , \end{aligned} \quad (54)$$

in which we have used (51) and the inclusions (52). Since $u \in A_0$, we conclude that

$$\|\mathcal{R}(u)\|_{H^{r_\alpha+m,p}(\mathbb{R}^n)} \leq C \delta \|\varphi\|_{L^\infty(\mathbb{R}^n)} \left(\|h\|_{L^p(\mathbb{R}^n)}^p + \tilde{C} \right) .$$

Hence, since the right side of the above inequality does not depend on u , there exists a sufficiently small δ such that for all $u \in A_0$

$$\|\mathcal{R}(u)\|_{H^{r_\alpha+m,p}(\mathbb{R}^n)} \leq 1 ,$$

and so the operator \mathcal{R} is well defined.

We now show that the operator \mathcal{R} has a fixed point on A_0 . We use the Schauder fixed point Theorem.

First we check continuity of \mathcal{R} . let us note that

$$\begin{aligned}
|V(x, u_1(x)) - V(x, u_2(x))| &= \left| \int_0^1 \frac{d}{dt} (V(x, tu_1(x) + (1-t)u_2(x))) dt \right| \\
&= \left| \int_0^1 D_y V(x, tu_1(x) + (1-t)u_2(x)) (u_1(x) - u_2(x)) dt \right| \\
&\leq |u_1(x) - u_2(x)| \int_0^1 |D_y V(x, tu_1(x) + (1-t)u_2(x))| dt \\
&\leq |u_1(x) - u_2(x)| C \int_0^1 |1 + |tu_1(x) + (1-t)u_2(x)||^\alpha dt \\
&\leq C |u_1(x) - u_2(x)| \int_0^1 |1 + t|u_1(x)|^\alpha + (1-t)|u_2(x)|^\alpha| dt \\
&\leq C |u_1(x) - u_2(x)| \int_0^1 (1 + |u_1(x)|^\alpha + |u_2(x)|^\alpha) dt \\
&= C |u_1(x) - u_2(x)| (1 + |u_1(x)|^\alpha + |u_2(x)|^\alpha).
\end{aligned}$$

We use this observation to estimate the difference $\varphi V(\cdot, u_1) - \varphi V(\cdot, u_2)$. We make use of the fact that φ has compact support, let us say $K \subset \mathbb{R}^n$.

$$\begin{aligned}
\|\varphi V(\cdot, u_1) - \varphi V(\cdot, u_2)\|_{L^p(\mathbb{R}^n)}^p &= \int_K |\varphi(x)|^p |V(x, u_1(x)) - V(x, u_2(x))|^p dx \\
&\leq C \|\varphi\|_{L^\infty(\mathbb{R}^n)} \int_K |u_1(x) - u_2(x)|^p (1 + |u_1(x)|^\alpha + |u_2(x)|^\alpha)^p dx \\
&\leq C \|\varphi\|_{L^\infty(\mathbb{R}^n)} \int_K |u_1(x) - u_2(x)|^p (1 + |u_1(x)|^{\alpha p} + |u_2(x)|^{\alpha p}) dx, \\
&\leq C \|\varphi\|_{L^\infty(\mathbb{R}^n)} \|u_1 - u_2\|_{L^p(K)}^p \left(1 + \|u_1\|_{L^\infty(\mathbb{R}^n)}^{\alpha p} + \|u_2\|_{L^\infty(\mathbb{R}^n)}^{\alpha p} \right),
\end{aligned}$$

in which C is a generic constant and we have used that $u_1, u_2 \in H^{r_\alpha+m,p}(\mathbb{R}^n)$ implies $u_1, u_2 \in L^\infty(\mathbb{R}^n)$, because of Prop 6.3 of [30] and the estimate on m appearing in the hypothesis of the theorem.

Hence, we obtain the following inequalities, in which we have used the continuous inclusions (52):

$$\begin{aligned}
\|\mathcal{R}(u_1) - \mathcal{R}(u_2)\|_{H^{r_\alpha+m,p}(\mathbb{R}^n)} &= \|w_1 - w_2\|_{H^{r_\alpha+m,p}(\mathbb{R}^n)} \leq C \|w_1 - w_2\|_{\mathcal{H}^{s,p}(a)} \\
&= C \|\delta \varphi (V(\cdot, u_1) - V(\cdot, u_2))\|_{L^p(\mathbb{R}^n)} \\
&\leq C \delta \|\varphi\|_{L^\infty(\mathbb{R}^n)} \|u_1 - u_2\|_{L^p(K)}^p \left(1 + \|u_1\|_{L^\infty(\mathbb{R}^n)}^{\alpha p} + \|u_2\|_{L^\infty(\mathbb{R}^n)}^{\alpha p} \right).
\end{aligned}$$

The second equality holds because of (45) and the fact that $v = w_1 - w_2$ is the solution to the linear problem $[1 + a(-\Delta)]^{s/2}(v) = V(\cdot, u_1) - V(\cdot, u_2)$.

Now we use that $\|u_1 - u_2\|_{L^p(K)}^p \leq C \|u_1 - u_2\|_{L^{\alpha p}(K)}^p$ —because $p < \alpha p$, since $\alpha > 1$ — and also that $\|u_1 - u_2\|_{L^{\alpha p}(K)}^p \leq \|u_1 - u_2\|_{L^{\alpha p}(\mathbb{R}^n)}^p$. Thus, the continuous

inclusions (52) and (50) imply

$$\|u_1 - u_2\|_{L^p(K)}^p \leq C_1 \|u_1 - u_2\|_{L^{\alpha p}(\mathbb{R}^n)}^p \leq C_2 \|u_1 - u_2\|_{H^{r_\alpha+m, p}(\mathbb{R}^n)}^p.$$

We conclude that if $\|u_1 - u_2\|_{H^{r_\alpha+m, p}(\mathbb{R}^n)}^p \rightarrow 0$ then $\|\mathcal{R}(u_1) - \mathcal{R}(u_2)\|_{H^{r_\alpha+m, p}(\mathbb{R}^n)} \rightarrow 0$. This fact shows the continuity of the operator \mathcal{R} .

Now we show that \mathcal{R} is a compact operator. Let us consider a bounded sequence $\{u_k\}$ in $H^{r_\alpha+m, p}(\mathbb{R}^n)$. We will show that the sequence $\{\mathcal{R}(u_k)\}$ has a convergent subsequence in $H^{r_\alpha+m, p}(\mathbb{R}^n)$. Since $\varphi \in C_0^\infty(\mathbb{R}^n)$, there exists $R > 0$ such that $\text{supp}(\varphi) \subset B(0, R)$. Now, for each $k \in \mathbb{N}$ let us define

$$g_k(x) = \begin{cases} \delta \varphi(x) V(x, u_k(x)) & \text{if } x \in B(0, R) \\ 0 & \text{if } x \in \mathbb{R}^n \setminus B(0, R). \end{cases}$$

We check that the sequence $\{g_k\}_{k \in \mathbb{N}}$ is bounded in $H^{1, p}(\mathbb{R}^n)$. We have,

$$\begin{aligned} \|g_k\|_{H^{1, p}(\mathbb{R}^n)} &= \delta \|\varphi V(\cdot, u_k)\|_{H^{1, p}(B)} \\ &= \delta \left(\|\varphi V(\cdot, u_k)\|_{L^p(B)} + \sum_{i=1}^n \|D_i [\varphi V(\cdot, u_k)]\|_{L^p(B)} \right)^{1/p}. \end{aligned}$$

Since for each k we have $u_k \in H^{r_\alpha+m, p}(\mathbb{R}^n)$, inequality (54) implies

$$\|\varphi V(\cdot, u_k)\|_{L^p(B)} < \infty,$$

and therefore we only need to show that $\sum_{i=1}^n \|D_i \varphi V(\cdot, u_k)\|_{L^p(B)} < \infty$. By the chain rule for weak derivatives, see [14], we have

$$\begin{aligned} D_i (\varphi(x) V(x, u_k(x))) &= D_i (\varphi(x)) V(x, u_k(x)) + \varphi(x) D_i (V(x, u_k(x))) \\ &= D_i (\varphi(x)) V(x, u_k(x)) + \\ &\quad \varphi(x) [D_i V(x, u_k(x)) + D_y V(x, u_k(x)) D_i u_k(x)] \end{aligned}$$

for $i = 1 \dots n$, and so,

$$\begin{aligned} &\|D_i \varphi V(\cdot, u_k)\|_{L^p(B)}^p \\ &= \int_B |D_i (\varphi(x)) V(x, u_k(x)) + \varphi(x) [D_i V(x, u_k(x)) + D_y V(x, u_k(x)) D_i u_k(x)]|^p dx \\ &\leq C(p) \left[\int_B |D_i (\varphi(x)) V(x, u_k(x))|^p dx + \int_B |\varphi(x)|^p |D_i V(x, u_k(x)) + D_y V(x, u_k(x)) D_i u_k(x)|^p dx \right] \\ &\leq C(p) \left[\|D_i V(\cdot, u_k)\|_{L^p(B)}^p \|V(\cdot, u_k)\|_{L^p(B)}^p + \right. \\ &\quad \left. C(p) \|\varphi\|_{L^p(B)}^p \left(\|D_i V(\cdot, u_k)\|_{L^p(B)}^p + \|D_y V(\cdot, u_k) D_i u_k\|_{L^p(B)}^p \right) \right]. \end{aligned}$$

Since (48) holds, we have

$$\begin{aligned}
\|D_i V(\cdot, u_k)\|_{L^p(B)}^p &= \int_B |D_i V(x, u_k(x))|^p dx \\
&\leq C \int_B |h(x)|^p + |u_k(x)|^{\alpha p} \\
&= C \left(\|h\|_{L^p(B)}^p + \|u_k\|_{L^{\alpha p}}^{\alpha p} \right) \\
&\leq C \left(\|h\|_{L^p(B)}^p + \tilde{C} \|u_k\|_{H^{r_\alpha+m,p}(B)}^\alpha \right), \tag{55}
\end{aligned}$$

On the other hand, we note that by (49)

$$\begin{aligned}
\|D_y V(\cdot, u_k) D_i u_k\|_{L^p(B)}^p &= \int_B |D_y V(x, u_k(x)) D_i u_k(x)|^p dx \\
&\leq C \int_B |(1 + |u_k(x)|^\alpha) D_i u_k(x)|^p dx \\
&= C \int_B |D_i u_k(x) + |u_k(x)|^\alpha D_i u_k(x)|^p dx \\
&\leq C(p) \left(\|D_i u_k\|_{L^p(B)} + \| |u_k|^{\alpha p} \|_{L^\infty(B)} \|D_i u_k\|_{L^p(B)}^p \right) \\
&= C(p) \|D_i u_k\|_{L^p(B)}^p (1 + \| |u_k|^{\alpha p} \|_{L^\infty(B)}). \tag{56}
\end{aligned}$$

Since the sequence $\{u_k\} \subset H^{r_\alpha+m,p}(\mathbb{R}^n)$ is bounded, there exists $M > 0$ so that

$$\|D_i u_k\|_{L^p(B)}^p \leq M$$

for all k . Moreover, by [31, Chp. 13, Sect. 6] we have that $H^{r_\alpha+m,p}(\mathbb{R}^n) \subset C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, and therefore we can conclude that $(1 + \| |u_k|^{\alpha p} \|_{L^\infty(B)}) < \infty$ uniformly in k . Hence by (56) we have

$$\|D_y V(\cdot, u_k) D_i u_k\|_{L^p(B)} < \infty.$$

Thus, we have proven that $\{g_k\}_{k \in \mathbb{N}}$ is a bounded sequence in $H^{1,p}(B)$. By the Rellich-Kondrachov theorem (see for instance [14, p. 274]) the embedding $H^{1,p}(B) \hookrightarrow L^p(B)$ is compact. Hence, there exists a subsequence $\{g_{k_i}\}$ of $\{g_k\}_{k \in \mathbb{N}}$ which converges in $L^p(B)$.

This fact allows us to show that the sequence $\{w_{k_i}\} = \mathcal{R}(u_{k_i})$ is a Cauchy sequence in $H^{r_\alpha+m,p}(\mathbb{R}^n)$. Indeed, we use the embedding $\mathcal{H}^{s,p}(a) \hookrightarrow H^{r_\alpha+m,p}(\mathbb{R}^n)$, see (52). Then, it follows that:

$$\begin{aligned}
\|w_{k_i} - w_{k_j}\|_{H^{r_\alpha+m,p}(\mathbb{R}^n)} &\leq C \|w_{k_i} - w_{k_j}\|_{\mathcal{H}^{s,p}(a)} \\
&= C \|\delta \varphi V(\cdot, u_{k_i}) - \delta \varphi V(\cdot, u_{k_j})\|_{L^p(\mathbb{R}^n)} \\
&\leq C \|g_{k_i} - g_{k_j}\|_{L^p(B)}.
\end{aligned}$$

Hence, the sequence $\{\mathcal{R}(u_{k_i})\}$ is convergent in the Banach space $H^{r_\alpha+m,p}(\mathbb{R}^n)$. We have proven that the operator \mathcal{R} is compact. By Schauder's theorem, there exists at least one fixed point u_0 of \mathcal{R} , and hence there exists a solution in $\mathcal{H}^{s,p}(a)$ to the equation (47). \square

We finish this section with an existence proof in the radial case. Our main technical references for this part of the paper is [23] and the later treatise [32]. Let us assume that t and p are real numbers such that $tp > n$. Then, Proposition 6.3 of [31] tells us that $H^{t,p}(\mathbb{R}^n) \subset C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. We define, after [23],

$$H_r^{t,p}(\mathbb{R}^n) = \{u \in H^{t,p}(\mathbb{R}^n) : u \text{ is spherically symmetric}\}.$$

Moreover, if $\beta sp > 4n$ and $a \in \mathcal{G}_s^\beta$, we define the following closed subspace of $\mathcal{H}^{s,p}(a)$:

$$\mathcal{H}_r^{s,p}(a) = \{u \in \mathcal{H}^{s,p}(a) : u \text{ is spherically symmetric}\}.$$

This definition makes sense because Theorem 3.11 implies that if $\beta sp > 4n$, then $\mathcal{H}^{s,p}(a)$ is contained in $C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. We also need the standard definition $L_r^p(\mathbb{R}^n) = \overline{C_{0,r}^\infty(\mathbb{R}^n)}^{L^p(\mathbb{R}^n)}$.

We begin by stating the following corollary of Theorem 4.1:

Corollary 4.4. *Let $\beta > 0, s \geq 0$ and $a \in \mathcal{G}_s^\beta$ be fixed. If $g \in L^p(\mathbb{R}^n)$ is a spherically symmetric function, then the solution to the linear equation (44) is also spherically symmetric.*

Proof. We have that the solution to equation (44) is given by

$$u_g = \mathcal{F}^{-1} \left(\frac{\mathcal{F}(g)}{(1 + a(|\xi|^2))^{s/2}} \right).$$

Since g is a spherically symmetric function, then $\mathcal{F}(g)$ is a spherically symmetric tempered distribution (see [22]), and therefore $\mathcal{F}(g)/(1 + a(|\xi|^2))^{s/2}$ is a spherically symmetric tempered distribution as well. Hence, u_g is spherically symmetric. \square

Theorem 4.5. *Let us assume that $\alpha > 1, \beta > 0$, suppose that $a \in \mathcal{G}_s^\beta$, and that $V(x, y)$ is spherically symmetric with respect to x . Assume also that there exist functions $h \in L^p(\mathbb{R}^n)$ and $g \in L^{\frac{\alpha p}{\alpha-1}}(\mathbb{R}^n)$ such that the following two inequalities hold:*

$$|V(x, y)| \leq C(|h(x)| + |y|^\alpha), \quad \left| \frac{\partial}{\partial y} V(x, y) \right| \leq C(|g(x)| + |y|^{\alpha-1}) \quad (57)$$

for some constant $C > 0$. Then, there exist $\epsilon > 0$ and $\rho_\epsilon > 0$ such that there is a radial (i.e. spherically symmetric) solution $u \in \mathcal{H}_r^{s,p}(a)$ to the equation

$$[1 + a(-\Delta)]^{s/2} u = V(\cdot, u) \quad (58)$$

with $\|u\|_{L_r^{\alpha p}(\mathbb{R}^n)} \leq \epsilon$ whenever $\|h\|_{L^p(\mathbb{R}^n)} < \rho_\epsilon$.

Proof. First of all we note that conditions (57) imply that if $u \in L_r^{\alpha p}(\mathbb{R}^n)$, then the function $V(\cdot, u) \in L^p(\mathbb{R}^n)$:

$$|V(x, u(x))|^p \leq C (|h(x)| + |u(x)|^\alpha)^p \leq C_1 (|h(x)|^p + |u(x)|^{\alpha p}) ,$$

and therefore

$$\|V(\cdot, u)\|_{L^p(\mathbb{R}^n)} \leq C_1 (\|h\|_{L^p(\mathbb{R}^n)} + \|u\|_{L_r^{\alpha p}(\mathbb{R}^n)}^\alpha) , \quad (59)$$

so that $V(\cdot, u) \in L^p(\mathbb{R}^n)$. Our hypotheses imply that, moreover, $V(\cdot, u)$ is radial for $u \in L_r^{\alpha p}(\mathbb{R}^n)$.

This observation allows us to define the function $\mathcal{G} : X_\epsilon \rightarrow L_r^{\alpha p}(\mathbb{R}^n)$, in which X_ϵ is the ball $X_\epsilon = \{u \in L_r^{\alpha p}(\mathbb{R}^n) : \|u\|_{L_r^{\alpha p}(\mathbb{R}^n)} \leq \epsilon\}$, as follows:

$$\mathcal{G}(u) = \tilde{u}$$

in which \tilde{u} is the unique solution to the linear equation

$$[1 + a(-\Delta)]^{s/2} \tilde{u} = V(\cdot, u) .$$

By Theorem 4.1 and Corollary 4.4, we conclude that $\mathcal{G}(u) = \tilde{u} \in \mathcal{H}_r^{s,p}(a)$ and that $\|\tilde{u}\|_{\mathcal{H}_r^{s,p}(a)} = \|V(\cdot, u)\|_{L^p(\mathbb{R}^n)}$. We check that the map \mathcal{G} is well defined. Indeed, let us set $r_\alpha = n(\alpha - 1)/(\alpha p)$ as in Theorem 4.3. Then, we have the inequalities

$$p < \alpha p < \frac{p n}{n - p r_\alpha}$$

for $n > 1$, and we can use the embedding theorems on radial functions appearing in Lions' paper [23], see also [32, Section 6.5.2]. We obtain, using also (52), the chain of continuous inclusions

$$\mathcal{H}_r^{s,p}(a) \hookrightarrow H_r^{r_\alpha, p}(\mathbb{R}^n) \hookrightarrow L_r^{\alpha p}(\mathbb{R}^n) , \quad (60)$$

in which “ \hookrightarrow ” denotes compact embedding. Thus, in particular, we have $\tilde{u} \in L_r^{\alpha p}(\mathbb{R}^n)$, as claimed.

Now we show that there exists $\epsilon > 0$ such that $\mathcal{G} : X_\epsilon \rightarrow X_\epsilon$. The continuous inclusions (60) imply that there exists $C_1 > 0$ such that

$$\|\mathcal{G}(u)\|_{L_r^{\alpha p}(\mathbb{R}^n)} \leq C_1 \|\mathcal{G}(u)\|_{\mathcal{H}_r^{s,p}(a)} = C_1 \|V(\cdot, u)\|_{L^p(\mathbb{R}^n)} , \quad (61)$$

in which we have used Theorem 4.1, and therefore by inequality (59) we obtain

$$\|\mathcal{G}(u)\|_{L^{\alpha p}(\mathbb{R}^n)} \leq C_1 C(\|h\|_{L^p(\mathbb{R}^n)} + \|u\|_{L^{\alpha p}(\mathbb{R}^n)}^\alpha) \leq C_1 C(\|h\|_{L^p(\mathbb{R}^n)} + \epsilon^\alpha). \quad (62)$$

We choose $\epsilon \geq (CC_1)^{1/(1-\alpha)}$ and $\|h\|_{L^p(\mathbb{R}^n)} < \rho_\epsilon := \epsilon/(CC_1) - \epsilon^\alpha$. Then we obtain $\|\mathcal{G}(u)\|_{L^{\alpha p}(\mathbb{R}^n)} \leq \epsilon$ and so $\mathcal{G} : X_\epsilon \rightarrow X_\epsilon$, as claimed.

As in Theorem 4.3, we plan to apply the Schauder fixed point theorem to the map \mathcal{G} . First, we show that the function \mathcal{G} is continuous:

Let us consider a sequence $\{u_n\} \subset X_\epsilon$ such that $u_n \rightarrow u$ in $L^{\alpha p}(\mathbb{R}^n)$. We set $\mathcal{G}(u_n) = \tilde{u}_n$ and $\mathcal{G}(u) = \tilde{u}$. We estimate $\|\tilde{u}_n - \tilde{u}\|_{L^{\alpha p}(\mathbb{R}^n)}$ as follows:

The continuous inclusions (60) yield

$$\|\tilde{u}_n - \tilde{u}\|_{L^{\alpha p}(\mathbb{R}^n)}^p \leq C \|\tilde{u}_n - \tilde{u}\|_{\mathcal{H}_r^{s,p}(a)}^p = C \|V(\cdot, u_n) - V(\cdot, u)\|_{L^p(\mathbb{R}^n)}, \quad (63)$$

and we can estimate this difference using the fundamental theorem of calculus and hypotheses (57):

$$\begin{aligned} & |V(x, u_n(x)) - V(x, u(x))| \\ &= \left| \int_0^1 \frac{d}{dt} [V(x, tu_n(x) + (1-t)u(x))] dt \right| \\ &= \left| (u_n(x) - u(x)) \int_0^1 \left(\frac{\partial}{\partial y} [V(x, tu_n(x) + (1-t)u(x))] \right) dt \right| \\ &\leq C |u_n(x) - u(x)| \int_0^1 (|g(x)| + |tu_n(x) + (1-t)u(x)|^{\alpha-1}) dt \\ &\leq C_1 |u_n(x) - u(x)| (|g(x)| + |u_n(x)|^{\alpha-1} + |u(x)|^{\alpha-1}) , \end{aligned}$$

and therefore by Hölder inequality we have

$$\begin{aligned} & \|V(\cdot, u_n) - V(\cdot, u)\|_{L^p(\mathbb{R}^n)}^p \leq \\ & C_1 \left(\int |u_n(x) - u(x)|^{\alpha p} \right)^{1/\alpha} \left(\int [|g(x)| + |u_n(x)|^{\alpha-1} + |u(x)|^{\alpha-1}]^{\alpha p/(\alpha-1)} \right)^{(\alpha-1)/(\alpha p)}. \end{aligned}$$

It follows that

$$\begin{aligned} & \|V(\cdot, u_n) - V(\cdot, u)\|_{L^p(\mathbb{R}^n)}^p \leq \\ & C_2 \|u_n - u\|_{L^{\alpha p}(\mathbb{R}^n)}^p \left[\int |g(x)|^{\alpha p/(\alpha-1)} dx + \int (|u_n(x)|^{\alpha p} + |u(x)|^{\alpha p}) dx \right]^{(\alpha-1)/(\alpha p)}. \end{aligned}$$

Since $u_n \rightarrow u$ in $L^{\alpha p}(\mathbb{R}^n)$ and $g \in L^{\frac{\alpha p}{\alpha-1}}(\mathbb{R}^n)$ by hypothesis, inequality (63) implies that $\tilde{u}_n \rightarrow \tilde{u}$ in $L^{\alpha p}(\mathbb{R}^n)$, so that \mathcal{G} is continuous, as claimed.

Now we prove that \mathcal{G} is compact. We use once more the inclusions (60). Let $(u_n)_{n \in \mathbb{N}} \subset X_\epsilon$ be a bounded sequence, so that $\|u_n\|_{L^{\alpha p}(\mathbb{R}^n)} \leq M$ for all $n \in \mathbb{N}$. We have,

$$\begin{aligned} \|\mathcal{G}(u_n)\|_{H_r^{r\alpha, p}(\mathbb{R}^n)} &\leq C \|\mathcal{G}(u_n)\|_{\mathcal{H}_r^{s, p}(a)} \\ &= C \|V(\cdot, u_n)\|_{L^p(\mathbb{R}^n)} \\ &\leq CC_1 \left(\|h\|_{L^p(\mathbb{R}^n)} + \|u_n\|_{L^{\alpha p}(\mathbb{R}^n)}^\alpha \right) , \end{aligned}$$

in which we have used (59) in the last inequality. Thus, the sequence $(\mathcal{G}(u_n))_{n \in \mathbb{N}}$ is bounded in $H_r^{r\alpha, p}(\mathbb{R}^n)$. Since the last continuous inclusion in (60) is compact, we conclude that this sequence has a convergent subsequence in X_ϵ with respect to the topology of $L^{\alpha p}(\mathbb{R}^n)$.

In conclusion, the map $\mathcal{G} : X_\epsilon \rightarrow X_\epsilon$ is compact and continuous. By Schauder's fixed point theorem, we have that \mathcal{G} possesses a fixed point. By Theorem 4.1 this fixed point belongs to $\mathcal{H}_r^{s, p}(a)$, and therefore, there exists a radial solution to the nonlinear equation (58) in $\mathcal{H}_r^{s, p}(a)$. \square

5 An Example: the fractional Laplace operator

We present some special cases of Theorem 4.5 on the existence of radial solutions, in the case when $a(\Delta) = (-\Delta)^{\alpha/2}$, $0 < \alpha < 1$. We are mainly interested in the fractional Allen-Cahn equation

$$(-\Delta)^{\alpha/2} u = -u + u^3 , \tag{64}$$

in the Benjamin-Ono equation

$$(-\Delta)^{\alpha/2} u = u^2 - u , \tag{65}$$

and in the non-linear equation

$$(-\Delta)^{\alpha/2} u + u - |u|^\beta u = 0 . \tag{66}$$

These equations have been much studied. Of the many papers considering the Allen-Cahn equation (64), we cite the recent works [8, 9] and [13]. The Benjamin-Ono equation (65) is studied (for $\alpha = 1$) for example in [1], in which they explain its relationship with the evolutionary version of the Benjamin-Ono equation of soliton theory. Finally, Equation (66) appears in the recent papers [15] and [16] as a general case of the fractional non-linear Schrödinger equation.

These three equations fall within our framework: we recall that $a(t) = (-t)^{\alpha/2}$ belongs to the class \mathcal{G}_s^α for all $s \geq 0$, as shown in Lemma 2.2, and therefore it makes sense to apply our foregoing results apply to equations of the form

$$[1 + (-\Delta)^{\alpha/2}]^{s/2} u = V(u) \quad (67)$$

for appropriate $V(u)$. We take $s = 2$ in Theorem 4.5 and state the following result on the existence of radial solutions in $L^p(\mathbb{R}^n)$ to Equations (64), (65) and (66):

Proposition 5.1. *Let $a(t) = (t)^{\alpha/2}$, $\delta > 1$, and assume that $V(u)$ is any of u^3 , u^2 or $|u|^\beta u$. There exists a spherically symmetric solution $u \in \mathcal{H}_r^{2,p}(a)$ to the equation*

$$[1 + (-\Delta)^{\alpha/2}]u = V(u) . \quad (68)$$

This $L^p(\mathbb{R}^n)$ -solution belongs to the radial fractional Sobolev space $H_r^{r_\delta,p}(\mathbb{R}^n)$ in which $r_\delta = n(\delta - 1)/(\delta p)$, and moreover, there exists $\epsilon > 0$ such that $u \in L_r^{\delta p}(\mathbb{R}^n)$ and $\|u\|_{L_r^{\delta p}(\mathbb{R}^n)} \leq \epsilon$.

Proof. The three alternatives for $V(u)$ appearing in the hypotheses of the theorem satisfy conditions (57) for appropriate functions $h(x)$ and $g(x)$ both different from zero. The theorem then follows from Theorem 4.5 and the continuous inclusions (60). \square

We finish this paper with an application of our theory to the stationary perturbed fractionary Allen-Cahn equation. The classical case is discussed for example in [24, 34, 35], and a recent application appears in [6].

Theorem 5.2. *Let $a(t) = \xi^2(t)^{\alpha/2}$ and $\delta > 1$, in which $\xi > 0$. Then, there exists a spherically symmetric solution $u \in \mathcal{H}_r^{2,p}(a)$ to the equation*

$$[1 + \xi^2(-\Delta)^{\alpha/2}]u = u^3 + \rho(|x|) , \quad (69)$$

in which $\rho(|x|)$ is a rapidly decreasing smooth radial positive function. This $L^p(\mathbb{R}^n)$ -solution u belongs to the radial fractional Sobolev space $H_r^{r_\delta,p}(\mathbb{R}^n)$ in which $r_\delta = n(\delta - 1)/(\delta p)$, and moreover, there exists $\epsilon > 0$ such that $u \in L_r^{\delta p}(\mathbb{R}^n)$ and $\|u\|_{L_r^{\delta p}(\mathbb{R}^n)} \leq \epsilon$.

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References

- [1] C.J. Amick and J.F. Toland, Uniqueness and related analytic properties for the Benjamin-Ono equation— a nonlinear Neumann problem in the plane. *Acta Math.*, 167 (1991), 107-126.
- [2] W. Arendt, C.J.K. Batty, M. Hieber and F. Neubrander *Vector-Valued Laplace Transforms and Cauchy Problems*, Birkhauser, 2011
- [3] N. Barnaby, A new formulation of the initial value problem for nonlocal theories. *Nuclear Physics B* , 845 2011
- [4] N. Barnaby and N. Kamran, Dynamics with infinitely many derivatives: the initial value problem. *J. High Energy Physics* 02, 40 pp., 2008
- [5] N. Barnaby and N. Kamran, Dynamics with infinitely many derivatives: variable coefficient equations. *J. High Energy Physics* 12, 27 pp, 2008
- [6] M. Beneš, V. Chalupecký and K. Mikula, Geometrical image segmentation by the Allen–Cahn equation. *Applied Numerical Mathematics* 51 (2004), 187–205.
- [7] M. Bravo, Nonlinear equations of infinite order defined by an elliptic symbol. *International Journal of Mathematics and Mathematical Sciences*, Article ID 656959, 7pp., 2014, 2014
- [8] X. Cabré and Y. Sire, Nonlinear equations for fractional Laplacians, I: Regularity, maximum principles, and Hamiltonian estimates. *Annales de l'Institut Henri Poincaré (C) Non Linear Analysis* 31 (2014), 23–53.
- [9] X. Cabré and Y. Sire, Nonlinear equations for fractional Laplacians II: existence, uniqueness, and qualitative properties of solutions. *Transactions AMS* 367 (2015), 911–941.
- [10] G. Calcagni and M. Montobbio and G. Nardelli, *Localization of nonlocal theories.*, Physics Letters B 3, 662, 2008
- [11] G. Calcagni and M. Montobbio and G. Nardelli, Route to nonlocal cosmology. *Physical Review D*, 12 , 76, 2007
- [12] M. Carlsson, H. Prado and E.G. Reyes, Differential equations with infinitely many derivatives and the Borel transform. *Annales Henri Poincaré* 17 Issue 8 (2016), 2049–2074.

- [13] E. Cinti, J. Davila and M. Del Pino, Solutions of the fractional Allen–Cahn equation which are invariant under screw motion. *J. London Math. Soc.* 94 (2016), 295–313.
- [14] L. Evans, *Partial Differential Equations*, American Mathematical Society., 1998
- [15] R.L. Frank and E. Lenzmann, Uniqueness of non-linear ground states for fractional Laplacians in \mathbb{R} , *Acta Math.* 210 (2013), 261–318.
- [16] R.L. Frank, E. Lenzmann and L. Silvestre, Uniqueness of Radial Solutions for the Fractional Laplacian. *Commun. Pur. Appl. Math.* 69 (2016), 1671–1726.
- [17] P. Górka and H. Prado and E.G. Reyes, On a general class of nonlocal equations, *Ann. Henri Poincare*, 947–966, 14, 2013
- [18] P. Górka and H. Prado and E.G. Reyes, Generalized euclidean bosonic string equations. In: Spectral analysis of quantum Hamiltonians, 147–169, Springer Basel AG, Oper. Theory Adv. Appl., 224, 224, 2012
- [19] P. Górka and H. Prado and E.G. Reyes, Nonlinear equations with infinitely many derivatives. *Complex Anal. Oper. Theory* 5 (2011), 313–323,
- [20] D. Guidetti, Vector valued Fourier multipliers and applications, *Bruno Pini Math. Anal. Seminar*, doi: <http://dx.doi.org/10.6092/issn.2240-2829/2231>, 2010
- [21] L. Hörmander, Estimates for translation invariant operators in L_p , *Acta Math.* 104 (1960), 93–140.
- [22] R. Kanwal, *Generalized Functions: Theory and Techniques*, Academic Press, 1983
- [23] P-L Lions, Symétrie et compacité dans les espaces de Sobolev, *Journal of Functional Analysis*, 315–334, 49, 1982
- [24] L. Mugnai and M. Röger, Convergence of Perturbed Allen-Cahn Equations to Forced Mean Curvature Flow. *Indiana University Mathematics Journal* Vol. 60, No. 1 (2011), 41–75.
- [25] M. Reed and B. Simon, *Methods of Modern Mathematical Physics Vol. I: Functional Analysis*. Academic Press., 1980
- [26] X. Saint-Raymond, *Elementary Introduction to the Theory of Pseudodifferential Operators*, CRC Press, 1991

- [27] Y. Sire and E. Valdinoci, Fractional Laplacian phase transitions and boundary reactions: a geometric inequality and a symmetry result, *J. Funct. Anal.*, 6, 1842-1864, 256, 2009
- [28] E. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press., 1970
- [29] E. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, 1971
- [30] M. Taylor, *Partial Differential Equations I. Basic Theory*, Springer, New York, USA, 2010
- [31] M. Taylor, *Partial Differential Equations III. Nonlinear Equations*, Springer, New York, USA, 2010
- [32] H. Triebel, *Theory of Function Spaces III*. Birkhauser Verlag, Basel-Boston-Berlin 2006
- [33] V. Vladimirov, The equation of the p -adic open string for the scalar tachyon field., *Investiya Mathematics*, 5, 487-512, 69, 2005
- [34] J. Wei and M. Winter, Stationary solutions for the Cahn-Hilliard equation. *Ann. Inst. Henri Poincare, Analyse non lineaire*, 15 (1998), 459–492.
- [35] J. Wei and M. Winter, On the Stationary Cahn-Hilliard Equation: Interior Spike Solutions. *J. Differential Equations* 148 (1998), 231–267.
- [36] M. W. Wong, M-Elliptic Pseudo-Differential operator on $L^p(\mathbb{R}^n)$, *Math. Nachr.*, 3, 319-326, 279, 2006